

Some Harder trigonometry problems

1. Let $\tan x \tan 2x + \tan 2x \tan 3x + \cdots + \tan 6x \tan 7x = 10$, find the value of $\frac{\tan 7x}{\tan x}$.

2. Let $\tan 2x = \frac{2t\sqrt{1-t^2}}{1-2t^2}$, and assume $0 < t < \frac{1}{\sqrt{2}}$. Find $\sin x$.

3. Compute $\sin \frac{\pi}{7} \sin \frac{2\pi}{7} \sin \frac{3\pi}{7}$.

4. (a) By using induction on $2(x^n + y^n) \geq (x^{n-1} + y^{n-1})(x + y)$, where $x, y \geq 0$.
 prove the Power Mean inequality: $\frac{x^n + y^n}{2} \geq \left(\frac{x+y}{2}\right)^n$.

 (b) Prove that $\frac{1}{2^{n-1}} \leq \sin^{2n} x + \cos^{2n} x \leq 1, \forall x \in \mathbf{R}$.

5. If $\frac{x}{\tan(\theta+\alpha)} = \frac{y}{\tan(\theta+\beta)} = \frac{z}{\tan(\theta+\gamma)}$, find the value of :

$$E = \left(\frac{x+y}{x-y}\right) \sin^2 (\alpha - \beta) + \left(\frac{y+z}{y-z}\right) \sin^2 (\beta - \gamma) + \left(\frac{z+x}{z-x}\right) \sin^2 (\gamma - \alpha)$$

6. Let $\begin{cases} \cos x - \cos y = a \\ \cos 3x - \cos 3y = b \end{cases}$
 Find the value of
 $E = \cos 2x + \cos 2y + 2 \cos x \cos y$

7. Let $T_n = \sin^n x + \cos^n x$
 Prove that $6T_{10} - 15T_8 + 10T_6 = 1$.

8. Let $(1 - k)\tan^2 \left(\frac{a}{2}\right) = (1 + k)\tan^2 \left(\frac{b}{2}\right)$, find the value of
 $E = k^2 + (1 + k \cos a)(1 - k \cos b)$.

9. Compute : $\sec^2 \frac{\pi}{7} + \sec^2 \frac{2\pi}{7} + \sec^2 \frac{3\pi}{7}$.

10. Find the sum of the series:
 $S = 1 + 2 \cos x + 2 \cos 2x + 2 \cos 3x + \cdots + 2 \cos nx$